Fractional Calculus of Mittag – Leffler type function using Pathway integral operator

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Abstract: The aim of the present paper is to evaluate new relations using Pathway integral operator onMittag - Leffler type functions with two Fractional orders. Each order plays an important rule while modeling for instance problems with two layers with different properties. The formulas established here are basic in nature and are likely to find useful applications in the field of science and engineering. Pathway integral operator generalizes the classical Riemann – Liouville fractional integration operator, and when $\alpha \to 1$ it reduces to the Laplace integral transform.

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1. Introduction and Preliminaries

Several definitions of operators of classical and generalized fractional calculus are already well known and widely used in the applications to mathematical model of fractional order. The most popular one, we are based on here, is the so-called Riemman – Liouville fractional integral (see for example the encyclopedia of fractional calculus by Samko – Kilbas – Merichev [16]). As operator of classical fractional calculus, also the Erdélyi – Kober operators are well known (cf. [16], Kiryakova [20]; Kober [11], Erdélyi [2], see also [17]). Here we introduce a fractional integration operator, which may be regarded as an extension of the left – sided Riemann – Liouville fractional integral operator. We propose some result for Pathway integral operator, including the Mittag – Leffler function, Mittag – Leffler type function, generalized function $G_{\rho,\delta,r}\left[a,z\right]$ defined in [14].

Let us recall the definition of left sided Riemann – Liouville fractional integral operator. Let$f(x) \in L(a,b)$, then

$$\left(I_{a+}^{\alpha}\psi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\psi(t)}{(x-t)^{1-\alpha}}\,dt,\ \left(\alpha \in C,\ \mathrm{Re}(\alpha) > 0\right) \qquad ...(1.1)$$

For more details, see [16], [20], [1], and other books on fractional calculus.

If $f(t)$ is replaced by $t^{\gamma} f(t)$in (1.1), the above operator turns out to be the Erdélyi – Kober fractional integral; if it is replaced by ${}_2F_1\left(\eta+\beta,-\gamma;\eta;1-\frac{t}{x}\right) f(t)$, then (1.1) takes the form of the Saigo hyper geometric fractional integral, see e.g. [15]:

$$\frac{\Gamma(\eta)}{x^{-\eta-\beta}} I_{0+}^{\eta,\beta,\gamma} f(x) = \int_0^x (x-t)^{\eta-1} {}_2F_1\left(\eta+\beta,-\gamma;\eta;1-\frac{t}{x}\right) f(t)\,\mathrm{d}t, \qquad \text{... (1.2)}$$

Many other operators of generalized fractional calculus can be obtained if on the place of $f(t)$one can use $\phi(t) f(t)$with a suitably chosen special function$\phi(t)$.

Definition 1.1: The **special functions** $G_{\rho,\eta,\gamma}[a, z]$

The special *G* and *R*-functions are defined by [8, p. 15, eqn. (101); 9, p.1, eqn. (1.2)]:

$$G_{\rho,\eta,r}[a, z] = z^{r\rho-\eta-1} \sum_{n=0}^{\infty} \frac{(r)_n (a z^{\rho})^n}{\Gamma(\rho n+\rho r-\eta)\, n!}, \ \mathrm{Re}(\rho r-\eta) > 0 \qquad \text{...(1.1.1)}$$

where $(r)_n$ is the Pochhammer symbol (cf.[13, p. 2 and p. 5]) :

$$(r)_n = \frac{\Gamma(r+n)}{\Gamma(r)},$$

$$(r)_0 = 1,\ (r)_n = (r)(r+1)\ldots(r+n-1),\ (n = 1,2,\ldots); \qquad \text{...(1.1.2)}$$

Definition 1.2: The generalized Mittag-Leffler function

In 1971, Prabhakar (1971) introduced the generalized Mittag-Leffler function $E_{\rho,\mu}^{\gamma}(z)$ (see [3], [14]):

$$E_{\rho,\mu}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\rho k+\mu)\, k!}, \qquad \text{...(1.2.1)}$$

$$\left(\rho,\ \mu,\ \gamma \in C,\ \mathrm{Re}(\rho) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0\right)$$

where at $\gamma = 1$, $E^{1}_{\rho,\mu}(z)$ coincides with the classical Mittag-Leffler function $E_{\rho,\mu}(z)$ and in particular $E_{1,1}(z) = e^{z}$ and when $\rho = 1$ it coincides with Kummer's confluent hypergeometric function $\phi(\gamma;\ \mu\ ; z)$ with the exactness to the constant multiplier $[\Gamma(\mu)]^{-1}$.

In 2007, Shukla and Prajapati (2007) (cf. [4]) introduced the function $E^{\gamma,q}_{\rho,\mu}(z)$, which is defined for $\rho,\ \mu,\ \gamma \in C,\ \mathrm{Re}(\rho) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0$ and $q\epsilon(0,1) \cup N$ as

$$E^{\gamma,q}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}\, z^{k}}{\Gamma(\rho k + \mu)\, k!}, \qquad \ldots(1.2.2)$$

In 2009, Tarik O. Salim (2009) (cf. [18]) introduced the function $E^{\gamma,\delta}_{\rho,\mu}(z)$, which is defined for $\rho,\ \mu,\ \gamma, \delta \in C;\ \mathrm{Re}(\rho) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\delta) > 0$ as

$$E^{\gamma,\delta}_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{k}\, z^{k}}{\Gamma(\rho k + \mu)\, (\delta)_{n}}, \qquad \ldots(1.2.3)$$

In 2012, a new generalization of Mittag – Leffler function was defined by Salim (2012) (cf. [19]) as

$$E^{\gamma,\delta,q}_{\rho,\mu,p}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk}\, z^{k}}{\Gamma(\rho k + \mu)\, (\delta)_{pk}}, \qquad \ldots(1.2.4)$$

Where $\rho,\ \mu,\ \gamma, \delta \in C;\ \min(\mathrm{Re}(\rho),\ \mathrm{Re}(\beta), \mathrm{Re}(\gamma), \mathrm{Re}(\delta)) > 0$.

Definition 1.3: **The Pathway fractional integration operator**

Let $f(x) \in L(a,b), \rho \in C, Re(\rho) > 0, a > 0$ and let us take a pathway parameter $\alpha < 1$. Then the pathway fractional integration operator, as an extension of (1.1), is defined and represented as follows (see [17, p. 239]):

$$\left(P_{0+}^{(\rho,\alpha,a)} f\right)(t) = t^{\rho} \int_{0}^{\frac{t}{a(1-\alpha)}} \left[1 - \frac{a(1-\alpha)\tau}{t}\right]^{\frac{\rho}{1-\alpha}} f(\tau)d\tau, \qquad \ldots (1.3.1)$$

Where $L(a,b)$ is the set of Lebesgue measurable functions defined on (a,b).

The pathway model is introduction by Mathai [5] and studied further by Mathai and Haubold [6], [7].

Definition 1.4: The generalized Wright's function is defined as follows (see, e.g. [10]):

$${}_p\psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots,(\alpha_p,A_p)\ ; \\ (\beta_1,B_1),\ldots,(\beta_q,B_q);\end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j n)} \cdot \frac{z^n}{n!} \qquad \ldots (1.4.1)$$

where the coefficients $A_1,\ldots,A_p$ and $B_1,\ldots,B_q$ are positive real numbers such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0$$

Required Result: The following formula is required (see [17, eq. (12)])

$$P_{0+}^{(\rho,\alpha,a)}\{t^{\beta-1}\} = \frac{t^{\rho+\beta}}{[a(1-\alpha)]^{\beta}} \frac{\Gamma(\beta)\Gamma\left(1+\frac{\rho}{1-\alpha}\right)}{\Gamma\left(\frac{\rho}{1-\alpha}+\beta+1\right)}, \qquad \ldots (1.4.2)$$

Where $\alpha < 1; Re(\rho) > 0; Re(\beta) > 0$.

2. **Pathway Fractional Integral of Generalized Functions**

In this section we establish some knew compositions of the pathway fractional integral given by (1.3.1) with Generalized function $G_{\eta,\mu,r}[c,z]$given by (1.1.1).

Theorem 2.1: Let$\eta, \mu, r, \rho \in C, a > 0, c \in R, \rho > 0, Re(\eta) > 0, Re(\mu) > 0, Re(r) > 0$ and $\alpha < 1$. Then we have the following relation:

$$P_{0+}^{(\rho,\alpha,a)}\{G_{\eta,\mu,r}[c,z]\}$$

$$= \frac{z^{\eta r-\mu+\rho}\,\Gamma\left(1+\frac{\rho}{1-\alpha}\right)}{[a(1-\alpha)]^{\eta r-\mu}\,\Gamma(\mathrm{r})}\;{}_1\psi_1\left[\begin{matrix}(r,1);\\ \left(\eta r-\mu+1+\dfrac{\rho}{1+\alpha},\eta\right);\end{matrix}\; c\left(\frac{z}{a(1-\alpha)}\right)^{\eta}\right] \qquad \ldots(2.1.1)$$

Proof. To prove the relation in (2.1.1), we denote left – hand side of the relation by $\Delta_1 i.e.$

$$\Delta_1 = P_{0+}^{(\rho,\alpha,a)}\{G_{\eta,\mu,r}[c,z]\}$$

Now using the definition (), we get

$$\Delta_1 = P_{0+}^{(\rho,\alpha,a)}\left\{z^{\eta r-\mu-1}\sum_{n=0}^{\infty}\frac{(r)_n(cz^{\eta})^n}{\Gamma(\eta n+\eta r-\mu)\,n!}\right\}$$

$$= \sum_{n=0}^{\infty}\frac{(r)_n c^n}{\Gamma(\eta n+\eta r-\mu)\,n!}P_{0+}^{(\rho,\alpha,a)}\{z^{\eta n+\eta r-\mu-1}\} \qquad \ldots(2.1.2)$$

By using the well – known relationship between the Beta function and the Gamma function (ef. [12, pp. 9-11] and [13, pp. 7-10]), it is easy to find the following formula (see also [17, eq. (12)]):

$$P_{0+}^{(\rho,\alpha,a)}\{t^{\beta-1}\} = \frac{t^{\rho+\beta}}{[a(1-\alpha)]^{\beta}}\frac{\Gamma(\beta)\Gamma\left(1+\frac{\rho}{1-\alpha}\right)}{\Gamma\left(\frac{\rho}{1-\alpha}+\beta+1\right)},$$

Here, using (1.4.2) with β replaced by $\eta n + \eta r - \mu$ to the pathway integral and after a simplification, we get

$$\Delta_1 = \sum_{n=0}^{\infty} \frac{(r)_n c^n \Gamma\left(1+\frac{\rho}{1-\alpha}\right)}{\Gamma\left(\eta n + \eta r - \mu + \frac{\rho}{1-\alpha} + 1\right) n!} \frac{z^{\eta n + \eta r + \rho - \mu}}{[a(1-\alpha)]^{\eta n + \eta r - \mu}}$$

$$= \frac{z^{\eta r + \rho - \mu}\Gamma\left(1+\frac{\rho}{1-\alpha}\right)}{[a(1-\alpha)]^{\eta r - \mu}} \sum_{n=0}^{\infty} \frac{\Gamma(r+n)}{\Gamma\left(\eta n + \eta r - \mu + \frac{\rho}{1-\alpha} + 1\right) n!} \frac{(cz^{\eta})^n}{[a(1-\alpha)]^{\eta n}\,\Gamma(r)} \quad \ldots(2.1.3)$$

Now in view of the result (1.4.1) therein, we at once arrive at the desired result in (2.1.1).

3. **Pathway Fractional Integral of Generalized Mittag – Leffler Functions**

Theorem 3.1: Let $\eta, \mu, \gamma, \rho, \lambda \in C, a > 0, c \in R, \rho > 0,$

$Re(\eta) > 0, Re(\mu) > 0, Re(\gamma) > 0, Re(\lambda) > 0, \beta > 0$ *and* $\alpha < 1$, q$\in (0,1) \cup N$. Then we have the following relation:

$$P_{0+}^{(\rho,\alpha,a)}\left\{z^{\lambda-1} E_{\eta,\mu}^{\gamma,q}(cz^{\beta})\right\}$$

$$= \frac{z^{\rho+\lambda}\,\Gamma\left(1+\frac{\rho}{1-\alpha}\right)}{[a(1-\alpha)]^{\lambda}\,\Gamma(\gamma)}\, {}_2\psi_2\left[\begin{matrix}(\gamma,q),(\lambda,\beta);\\ (\mu,\eta),\left(1+\lambda+\frac{\rho}{1+\alpha},\beta\right);\end{matrix} c\left(\frac{z}{a(1-\alpha)}\right)^{\beta}\right] \quad \ldots(3.1.1)$$

Proof. To prove the relation in (3.1.1), we denote left – hand side of the relation by $\Delta_2 i.e.$

$$\Delta_2 = P_{0+}^{(\rho,\alpha,a)}\left\{z^{\lambda-1} E_{\eta,\mu}^{\gamma,q}(cz^{\beta})\right\}$$

Now using the definition (1.2.2), we get

$$\Delta_2 = P_{0+}^{(\rho,\alpha,a)}\left\{z^{\lambda-1} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(cz^{\beta})^n}{\Gamma(\eta n + \mu)\, n!}\right\}$$

$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\, c^n}{\Gamma(\eta n + \mu)\; n!} P_{0+}^{(\rho,\alpha,a)}\{z^{\beta n+\lambda-1}\}$$

By using the well – known relationship between the Beta function and the Gamma function (ef. [12, pp. 9-11] and [13, pp. 7-10]), it is easy to find the following formula (see also [17, eq. (12)]), i.e.:

$$P_{0+}^{(\rho,\alpha,a)}\{t^{\beta-1}\} = \frac{t^{\rho+\beta}}{[a(1-\alpha)]^{\beta}} \frac{\Gamma(\beta)\Gamma\left(1+\frac{\rho}{1-\alpha}\right)}{\Gamma\left(\frac{\rho}{1-\alpha}+\beta+1\right)},$$

Here, using (1.4.2) with β replaced by $\beta n + \lambda$ to the pathway integral and after a simplification, we get

$$\Delta_2 = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}\, c^n\, \Gamma\left(1+\frac{\rho}{1-\alpha}\right)\Gamma(\beta n+\lambda)}{\Gamma(\eta n+\mu)\Gamma\left(\beta n+\lambda+\frac{\rho}{1-\alpha}+1\right)\; n!} \frac{z^{\beta n+\rho+\lambda}}{[a(1-\alpha)]^{\beta n+\lambda}}$$

$$= \frac{z^{\rho+\lambda}\Gamma\left(1+\frac{\rho}{1-\alpha}\right)}{[a(1-\alpha)]^{\lambda}\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+qn)\Gamma(\lambda+\beta n)}{\Gamma(\mu+\eta n)\Gamma\left(\beta n+\lambda+\frac{\rho}{1-\alpha}+1\right)\; n!} \frac{(cz^{\beta})^n}{[a(1-\alpha)]^{\beta n}}$$

Now in view of the result (1.4.1) therein, we at once arrive at the desired result in (3.1.1).

Theorem 3.2: Let $\eta, \mu, \gamma, \delta, \rho, \lambda \in C, a > 0, c \in R, \rho > 0,$

$Re(\eta) > 0, Re(\mu) > 0, Re(\gamma) > 0, Re(\delta) > 0, Re(\lambda) > 0, \beta > 0$ and $\alpha < 1$. Then we have the following relation:

$$P_{0+}^{(\rho,\alpha,a)}\{z^{\lambda-1} E_{\eta,\mu}^{\gamma,\delta}(cz^{\beta})\}$$

$$= \frac{z^{\rho+\lambda}\,\Gamma\left(1+\frac{\rho}{1-\alpha}\right)\Gamma(\delta)}{[a(1-\alpha)]^{\lambda}\,\Gamma(\gamma)}\, {}_3\psi_3\left[\begin{matrix}(\gamma,1),(\lambda,\beta),(1,1); \\ (\mu,\eta),\left(1+\lambda+\frac{\rho}{1+\alpha},\beta\right),(\delta,1);\end{matrix}\; c\left(\frac{z}{a(1-\alpha)}\right)^{\beta}\right]$$

…(3.2.1)

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Proof. To prove the relation in (3.2.1), we denote left – hand side of the relation by $\Delta_3 i.e.$

$$\Delta_3 = P_{0+}^{(\rho,\alpha,a)}\{z^{\lambda-1}E_{\eta,\mu}^{\gamma,\delta}\ (cz^{\beta})\}$$

Now using the definition (1.2.3), we get

$$\Delta_2 = P_{0+}^{(\rho,\alpha,a)}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n (cz^{\beta})^n}{\Gamma(\eta n+\mu)(\delta)_n}\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_n c^n}{\Gamma(\eta n+\mu)(\delta)_n}P_{0+}^{(\rho,\alpha,a)}\{z^{\beta n+\lambda-1}\}$$

By using the well – known relationship between the Beta function and the Gamma function (ef. [12, pp. 9-11] and [13, pp. 7-10]), it is easy to find the following formula (see also [17, eq. (12)]), i.e.:

$$P_{0+}^{(\rho,\alpha,a)}\{t^{\beta-1}\} = \frac{t^{\rho+\beta}}{[a(1-\alpha)]^{\beta}}\frac{\Gamma(\beta)\Gamma\left(1+\frac{\rho}{1-\alpha}\right)}{\Gamma\left(\frac{\rho}{1-\alpha}+\beta+1\right)},$$

Here, using (1.4.2) with β replaced by $\beta n+\lambda$ to the pathway integral and after a simplification, we get

$$\Delta_2 = \sum_{n=0}^{\infty}\frac{(\gamma)_n c^n\Gamma\left(1+\frac{\rho}{1-\alpha}\right)\Gamma(\beta n+\lambda)}{\Gamma(\eta n+\mu)\,\Gamma\left(\beta n+\lambda+\frac{\rho}{1-\alpha}+1\right)(\delta)_n}\frac{z^{\beta n+\rho+\lambda}}{[a(1-\alpha)]^{\beta n+\lambda}}$$

$$=\frac{z^{\rho+\lambda}\Gamma\left(1+\frac{\rho}{1-\alpha}\right)\Gamma(\delta)}{[a(1-\alpha)]^{\lambda}\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)\Gamma(\lambda+\beta n)\Gamma(1+n)}{\Gamma(\delta+n)\Gamma(\mu+\eta n)\,\Gamma\left(\beta n+\lambda+\frac{\rho}{1-\alpha}+1\right)\,n!}\frac{\left(cz^{\beta}\right)^n}{[a(1-\alpha)]^{\beta n}}$$

Now in view of the result (1.4.1) therein, we at once arrive at the desired result in (3.2.1).

Theorem 3.3: Let $\eta,\mu,\gamma,\delta,\rho,\lambda \in C, a>0, c\in R, \rho>0,$

$\min(Re(\eta)>0, Re(\mu)>0, Re(\gamma)>0, Re(\delta)>0, Re(\lambda)>0)\,,\beta>0$ and $\alpha<1.$

Then we have the following relation:

$$P_{0+}^{(\rho,\alpha,a)}\left\{z^{\lambda-1}E_{\eta,\mu,p}^{\gamma,\delta,q}\ (cz^{\beta})\right\}$$

$$= \frac{z^{\rho+\lambda}\ \Gamma\left(1+\frac{\rho}{1-\alpha}\right)\Gamma(\delta)}{[a(1-\alpha)]^{\lambda}\ \Gamma(\gamma)}\ {}_3\psi_3\left[\begin{matrix}(\gamma,q),(\lambda,\beta),(1,1);\\ (\mu,\eta),\left(1+\lambda+\dfrac{\rho}{1+\alpha},\beta\right),(\delta,p);\end{matrix}\ c\left(\frac{z}{a(1-\alpha)}\right)^{\beta}\right] \quad \ldots(3.3.1)$$

Proof. To prove the relation in (3.3.1), we denote left – hand side of the relation by $\Delta_3 i.e.$

$$\Delta_3 = P_{0+}^{(\rho,\alpha,a)}\left\{z^{\lambda-1}E_{\eta,\mu,p}^{\gamma,\delta,q}\ (cz^{\beta})\right\}$$

Now using the definition (1.2.4), we get

$$\Delta_3 = P_{0+}^{(\rho,\alpha,a)}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(cz^{\beta})^{n}}{\Gamma(\eta n+\mu)(\delta)_{pn}}\right\}$$

$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}c^{n}}{\Gamma(\eta n+\mu)(\delta)_{pn}}P_{0+}^{(\rho,\alpha,a)}\left\{z^{\beta n+\lambda-1}\right\}$$

By using the well – known relationship between the Beta function and the Gamma function (ef. [12, pp. 9-11] and [13, pp. 7-10]), it is easy to find the following formula (see also [17, eq. (12)]), i.e.:

$$P_{0+}^{(\rho,\alpha,a)}\left\{t^{\beta-1}\right\} = \frac{t^{\rho+\beta}}{[a(1-\alpha)]^{\beta}}\frac{\Gamma(\beta)\Gamma\left(1+\frac{\rho}{1-\alpha}\right)}{\Gamma\left(\frac{\rho}{1-\alpha}+\beta+1\right)},$$

Here, using (1.4.2) with β replaced by $\beta n+\lambda$ to the pathway integral and after a simplification, we get

$$\Delta_2 = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}c^{n}\Gamma\left(1+\frac{\rho}{1-\alpha}\right)\Gamma(\beta n+\lambda)}{\Gamma(\eta n+\mu)\ \Gamma\left(\beta n+\lambda+\frac{\rho}{1-\alpha}+1\right)(\delta)_{pn}}\frac{z^{\beta n+\rho+\lambda}}{[a(1-\alpha)]^{\beta n+\lambda}}$$

$$= \frac{z^{\rho+\lambda}\Gamma\left(1+\frac{\rho}{1-\alpha}\right)\Gamma(\delta)}{[a(1-\alpha)]^{\lambda}\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\Gamma(\lambda+\beta n)\Gamma(1+n)}{\Gamma(\delta+pn)\Gamma(\mu+\eta n)\ \Gamma\left(\beta n+\lambda+\frac{\rho}{1-\alpha}+1\right)\ n!}\frac{(cz^{\beta})^{n}}{[a(1-\alpha)]^{\beta n}}$$

Now in view of the result (1.4.1) therein, we at once arrive at the desired result in (3.3.1).

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